# MATH4210: Financial Mathematics Tutorial 8

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# PDEs

$u(t,x) = E[\cancel{\phi} e^{-r(T-t)} f(S_T) \mid S_t = x]$

## Question

*Assume* $u(t,x)$ *satisfies* $dS_t = \mu S_t dt + \sigma S_t dB_t$, then

$$\begin{cases} \partial_t u + \frac{\sigma^2}{2} x^2 \partial_x^2 u + rx\partial_x u - ru = 0 \\ u(T,x) = g(x) \end{cases}$$

$f(x)$

*Compute* $u(0, s_0)$

$u(t,x) = E[e^{-r(T-t)} g(S_T) \mid S_t = x]$

$S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T}$

$\Rightarrow u(0,s_0) = E[e^{-rT} g(s_0 e^{(\mu-\frac{1}{2}\sigma^2)T + \sigma B_T}) \mid S_0 = s_0]$

# Option Pricing of Continuous Market Models

$= E[e^{-rT} g(S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma B_T})]$

## Question

*We consider an Asian option with payoff*

$$g(S.) = \int_0^T S_u du$$

$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t}$

*under Black Scholes setting. Define*

$$V_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} g(S.)|\mathcal{F}_t].$$

$\sigma(S_\theta)_{0\le\theta\le t}$

*Show that*

$$\begin{cases} V_t = \int_0^t S_u du + (T-t)S_t & \text{if } r = 0 \\ V_t = e^{-r(T-t)} \int_0^t S_u du + \frac{1}{r}(1 - e^{-r(T-t)})S_t & \text{if } r > 0 \end{cases}$$

$$v_t = E^Q\Big[\underbrace{e^{-r(T-t)}\int_0^T S_u\,du}\Big|\underbrace{\mathcal{F}_t}\Big]$$

$$= E^Q\Big[e^{-r(T-t)}\Big(\int_0^t S_u\,du + \int_t^T S_u\,du\Big)\Big|\mathcal{F}_t\Big]$$

$$= E^Q\Big[e^{-r(T-t)}\int_0^t S_u\,du\Big|\mathcal{F}_t\Big] + E^Q\Big[e^{-r(T-t)}\int_t^T S_u\,du\Big|\mathcal{F}_t\Big]$$

$$= e^{-r(T-t)}\int_0^t \underbrace{E^Q[S_u|\mathcal{F}_t]}_{\substack{u\in[0,t]\\ S_u}}\,du + e^{-r(T-t)}\int_t^T \underbrace{E^Q[S_u|\mathcal{F}_t]}\,du$$

$$= E^Q\Big[S_t e^{(r-\frac{1}{2}\sigma^2)(u-t)+\sigma(B_u-B_t)}\Big|\mathcal{F}_t\Big]$$

$$= S_t e^{(r-\frac{1}{2}\sigma^2)(u-t)} E^Q\Big[e^{\sigma(B_u-B_t)}\Big|\mathcal{F}_t\Big]$$

($B_u-B_t$ is indepent of $\mathcal{F}_t$)

$$= S_t e^{(r-\frac{1}{2}\sigma^2)(u-t)} \underbrace{E^Q\Big[e^{\sigma(B_u-B_t)}\Big]}$$

$\sigma(B_u-B_t)\sim N(0,\sigma^2(u-t))$;

$E(e^X) = e^{\mu+\frac{1}{2}\sigma^2}$ if $X\sim N(\mu,\sigma^2)$

$$= S_t\cdot e^{(r-\frac{1}{2}\sigma^2)(u-t)}\, e^{0+\frac{1}{2}\sigma^2(u-t)}$$

$$= S_t e^{r(u-t)}$$

$$= e^{-r(T-t)}\int_0^t S_u\,du + e^{-r(T-t)}\int_t^T S_t e^{r(u-t)}\,du$$

① $r=0$

$$= \cancel{e^{-r(T-t)}}\int_0^t S_u\,du + \cancel{e^{-r(T-t)}}\int_t^T S_t\,du$$

$$= \int_0^t S_u\,du + S_t(T-t)$$

② $r>0$

$$= e^{-r(T-t)}\int_0^t S_u\,du + S_t e^{-r(T-t)}\underbrace{\int_t^T e^{r(u-t)}\,du}$$

$$\Big(\frac{e^{r(u-t)}}{r}\Big|_t^T\Big)$$

$$= e^{-r(T-t)}\int_0^t S_u\,du + S_t\cdot e^{-r(T-t)}\cdot\frac{1}{r}\big(e^{r(T-t)}-1\big) = \frac{1}{r}\big(e^{r(T-t)}\ldots$$

$r(e \ldots -1))$

$= e^{-r(T-t)} \int_0^t S_u du + \frac{S_t}{r}(1 - e^{-r(T-t)})$.

$dX_t =$

$dy = y \cdot dt + y \cdot dq(t)$

$\frac{dy}{y} = d(\ln y) = dt + q(t)$

## Question

*Consider the Black-Scholes SDE, for $\mu, \sigma$ constants:*

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

$\frac{dS_t}{\sigma S_t} = d(\ln S_t)$

(a) $d(\ln S_t) = \left(\frac{1}{S_t} \cdot \mu S_t + \frac{(-1) \cdot S_t^{-2}}{2} \cdot \sigma^2 \cdot S_t^2\right) dt + \left(\frac{1}{S_t} \cdot \sigma \cdot S_t \cdot dB_t\right)$

*(a). Solve the SDE.*

*(b). Find* $\mathbb{E}\left[\frac{S_{t+\Delta t}}{S_t}\right]$ *and* $\mathbb{E}\left[\left(\frac{S_{t+\Delta t}}{S_t}\right)^2\right]$

(Itô's formula). $f(B_t)$.

$= (\mu - \frac{\sigma^2}{2}) dt + \sigma dB_t$

$dX_t = b dt + \sigma dB_t$ $(b=0, \sigma=1, X_t = B_t)$

$\ln S_t - \ln S_0 = \int_0^t (\mu - \frac{\sigma^2}{2}) dt + \int_0^t \sigma dB_t$

① $f(X_t)$

$\sigma^2 dt$

$df(X_t) = f'(X_t) dX_t + \frac{f''(X_t)}{2}(dX_t)^2 = f'(X_t) dX_t + \frac{f''(X_t)}{2} \cdot \sigma^2 dt = f'(X_t)(b dt + \sigma dB_t) + \frac{f''(X_t)}{2}$

$$(dX_t)^2 = \cancel{b^2 \cdot (dt)^2}_{o(t^2)} + \boxed{\sigma^2 \cdot (dB_t)^2} + b\sigma \cdot \underbrace{dB_t\, dt}_{o(t^{1.5})}$$

$dB_t \sim o(t^{\frac{1}{2}})$

$dt \sim o(t)$

$(dB_t)^2 \approx dt$

$$= \left(f'(X_t) \cdot b + \frac{f''(X_t)}{2} \cdot \sigma^2\right) dt + f'(X_t) \cdot \sigma\, dB_t$$

$$\Rightarrow \ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t$$

$$\Rightarrow S_t = e^{\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t}$$

$$= S_0 \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t}$$

② $X_t Y_t$

$$d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t$$

(b) $$E\left(\frac{S_{t+\Delta t}}{S_t}\right) = E\left(\frac{S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)(t+\Delta t) + \sigma B_{t+\Delta t}}}{S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t}}\right) = E\left(e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma (B_{t+\Delta t} - B_t)}\right)$$

$$= e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t} \cdot E\left(e^{\sigma (B_{t+\Delta t} - B_t)}\right)$$

$$= e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t} \cdot e^{0 + \frac{1}{2}(\sigma^2 \cdot \Delta t)}$$

$$= e^{\mu \Delta t}$$

$$E\left(\left(\frac{S_{t+\Delta t}}{S_t}\right)^2\right) = E\left(\frac{S_0^2 \cdot e^{2\left(\mu - \frac{\sigma^2}{2}\right)(t+\Delta t) + 2\sigma B_{t+\Delta t}}}{S_0^2 \cdot e^{2\left(\mu - \frac{\sigma^2}{2}\right) t + 2\sigma B_t}}\right)$$

$$= E\left(e^{2\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \underbrace{2\sigma (B_{t+\Delta t} - B_t)}_{\sim N(0,\, 4\sigma^2 \Delta t)}}\right)$$

$$= e^{2\left(\mu - \frac{\sigma^2}{2}\right)\Delta t} \cdot e^{0 + \frac{1}{2} 4\sigma^2 \Delta t}$$

$$= e^{2\mu \Delta t + \sigma^2 \Delta t}$$

$dy = \cancel{p(x)} \cdot p(t) \cdot y\,dt + q(t) \Rightarrow y(t) = e^{\int_0^t p(s)ds}\left(y_0 + \int_0^t e^{-\int_0^s p_u du} q(s) \cdot ds\right)$

$X_t = e^{\mu t}\left( \cdots \right)$

## Question

*Solve the following SDEs, given* $\mu, \sigma$, *a constants:*

*(a). (Ornstein-Uhlenbeck)*

$d(e^{-\mu t} \cdot X_t)$

$$dX_t = \mu X_t dt + \sigma dB_t.$$

*(b). (Vasicek)*

$$dr_t = (a - r_t)dt + \sigma dB_t$$

$d(e^t r_t)$

$d(e^{-\mu t} X_t) = -\mu e^{-\mu t} dt \cdot X_t + e^{-\mu t} \cdot dX_t$

$= -\mu e^{-\mu t} \cdot X_t dt + e^{-\mu t} (\mu X_t dt + \sigma dB_t)$

$= \cancel{-\mu e^{-\mu t} X_t dt} + \cancel{\mu \cdot e^{-\mu t} X_t dt} + \sigma \cdot e^{-\mu t} dB_t$

$= \sigma \cdot e^{-\mu t} dB_t$

$$e^{-\mu t}\underbrace{X_t}_{} - X_0 = \int_0^t \sigma \cdot e^{-\mu s} dB_s$$

$$\Rightarrow X_t = e^{\mu t}\left(X_0 + \int_0^t \sigma \cdot e^{-\mu s} dB_s\right)$$

$$\begin{aligned} d(e^t r_t) &= e^t dt \cdot r_t + e^t dr_t \\ &= r_t \cdot e^t dt + e^t \left( (a - r_t) dt + \sigma dB_t \right) \\ &= a \cdot e^t dt + \sigma \cdot e^t dB_t \end{aligned}$$

$$e^t r_t - r_0 = \underbrace{\int_0^t a e^s ds}_{a(e^t - 1)} + \int_0^t \sigma \cdot e^s dB_s$$

$$\Rightarrow \quad r_t = e^{-t}\left(r_0 + a(e^t - 1) + \int_0^t \sigma e^s dB_s\right)$$